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A NOTE ON THE PROBLEM OF PRESCRIBING GAUSSIAN CURVATURE ON SURFACES

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ABSTRACT. The problem of existence of conformal metrics with Gaussian curvature equal to a given function K on a compact Riemannian 2-manifold M of negative Euler characteristic is studied. Let K_0 be any nonconstant function on M with $\max K_0 = 0$, and let $K_\lambda = K_0 + \lambda$. It is proved that there exists a $\lambda^* > 0$ such that the problem has a solution for $K = K_\lambda$ iff $\lambda \in (-\infty, \lambda^*]$. Moreover, if $\lambda \in (0, \lambda^*)$, then the problem has at least 2 solutions.

Let M be a closed 2-dimensional smooth manifold and g be a Riemannian metric on M. Let k denote the Gaussian curvature of g. If $g' = e^{2u}g$ is another Riemannian metric conformal to g, and has Gaussian curvature k', then it is well known that

$$k' = e^{-2u}(k - \Delta u),$$

where Δ is the Laplacian of g. Given a function $K \in C^\infty(M)$, the problem of prescribing Gaussian curvature asks whether one can find $u \in C^\infty(M)$ such that the metric $g' = e^{2u}g$ has the given K as its Gaussian curvature. Obviously, this is equivalent to the problem of solvability of the following elliptic equation

$$\Delta u - k + Ke^{2u} = 0, \quad \text{on } M. \tag{1}$$

If u is a solution of (1), then we have by integrating (1)

$$\int_M Ke^{2u} dv = \int_M k dv,$$

where dv is the area element with respect to the metric g. It follows from the Gauss-Bonnet formula that

$$\int_M ke^{2u} dv = 2\pi\chi(M), \tag{2}$$

where $\chi(M)$ is the Euler characteristic of M. Note that (2) poses restrictions on the given function K for the solvability of (1), according to the sign of $\chi(M)$.

If $\chi(M) = 0$, the problem of the solvability of (1) has been completely resolved. (See [K–W].) If $\chi(M) > 0$, then M is either RP^2 (the real projective

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plane) or S^2 (the 2-sphere). While the case where $M = RP^2$ has been well understood (see [M], [A]), the case where $M = S^2$ is much more complicated. Many authors have studied the problem on S^2 with its standard metric, known as Nirenberg problem (see e.g. [C–D], [C–Y1, 2], [C–L]).

In this note we consider only the case where $\chi(M) < 0$; in other words,

$$\int_M k dv < 0 .$$

This case has been studied by Kazdan and Warner in [K–W] using the method of super- and sub-solutions for second order elliptic equations. The following are some facts derived by them.

Fact (i). One can always find an arbitrarily negative subsolution φ for equation (1). Indeed, such a subsolution can be of the form $\varphi_c = f - c$, where f is a solution to the equation $\Delta f = k - \overline{k}$ with $\overline{k}$ being the mean value of k, and c is any sufficiently large number. Therefore, to solve (1) one needs only to find a supersolution ψ for (1).

Fact (ii). Let $K_1 \geq K_2$ are two smooth functions on M. Suppose that (1) has a solution u_1 for $K = K_1$. Then, since u_1 is a supersolution for (1) with $K = K_2$ as can be easily checked, we see that (1) is solvable for $K = K_2$ by Fact (i).

Fact (iii). It is easy to see from (2) that a necessary condition for (1) to be solvable is that the function K is negative somewhere on M. On the other hand, if $K \leq 0$, then one can find a supersolution for (1). It follows from Fact (i) that (1) has a solution provided $K \leq 0$. Moreover, in such a case, one can show that the solutions of (1) are unique.

In view of Fact (iii), we are only interested in the case where the function K changes sign. From now on, we assume that $K_0 \in C^\infty(M)$ is a nonconstant function which satisfies

$$\operatorname{Max}_{x \in M} K_0(x) = 0 \tag{3}$$

and let $K_\lambda = K_0 + \lambda$, where λ is a real number. Consider the family of equations

$$\Delta u - k + K_\lambda e^{2u} = 0 . \tag*{$(1)_\lambda$}$$

By Fact (iii), $(1)_\lambda$ has a unique solution u_λ for $\lambda \leq 0$. On the other hand, for the solution u_0 of $(1)_0$, the variational equation

$$\Delta v + 2K_0 e^{2u_0} v = 0$$

has only a trivial solution $v \equiv 0$, since $K_0 \leq 0$ and $K_0 \not\equiv 0$. It follows from the implicit function theorem that $(1)_\lambda$ has a solution for sufficiently small $\lambda > 0$. So we have

Lemma 1. *There exists a $\lambda^* > 0$ such that $(1)_\lambda$ is solvable for all $\lambda < \lambda^*$, and it has no solutions for $\lambda > \lambda^*$.*

Proof. Let λ^* be the supremum of all λ for which $(1)_\lambda$ has a solution. We have known that $\lambda^* < 0$, and $\lambda^* < -\inf_M K_0$ by (iii). It follows from Fact (ii) that λ^* has the claimed property.

Our main result is as follows.

Theorem. *Let $K_0 \in C^\infty(M)$ be any nonconstant function satisfying* (3), *and let* $K_\lambda = K_0 + \lambda$. *Then there exists a* $\lambda^* > 0$ *such that* (a) $(1)_\lambda$ *has a unique solution for* $\lambda \le 0$; (b) $(1)_\lambda$ *has at least two solutions if* $0 < \lambda < \lambda^*$; *and* (c) $(1)_{\lambda^*}$ *has at least one solution.*

Remark. If we set

$$S = \{K \in C^\infty(M) \colon (1) \text{ is solvable}\},$$

then the Theorem implies that the set $S \cup \{0\}$ is closed in C^0 topology. Indeed, let $\{K_i\} \subset S$ be a sequence such that $K_i \to K \in C^\infty(M)\backslash\{0\}$. Then for any $\varepsilon > 0$ we can find K_i such that $K - \varepsilon \le K_i$, and this shows that $K - \varepsilon \in S$ for any $\varepsilon > 0$. It follows from (c) of the Theorem that $K \in S$.

Now we turn to the proof of the Theorem. It is clear that conclusion (a) follows from Fact (iii). Hence we need only prove (b) and (c).

Proof of (b) *of the Theorem.* Note that $(1)_\lambda$ is the Euler-Lagrange equation of the functional

$$I_\lambda(u) = \int_M (|\nabla u|^2 + 2ku - K_\lambda e^{2u})\, dv .$$

We are to apply variational methods (see [C]) to obtain multiple critical points for I_λ, which correspond to solutions of $(1)_\lambda$, for $\lambda \in (0, \lambda^*)$. Fixing any $\lambda \in (0, \lambda^*)$, we choose a $\lambda_1 \in (\lambda, \lambda^*)$. Let ψ be a solution of $(1)_{\lambda_1}$. Then ψ is a super-solution for the equation $(1)_\lambda$. By Fact (i), we can find a sub-solution φ for $(1)_\lambda$ such that $\varphi < \psi$ on M. Let $[\varphi, \psi]$ be the order interval defined by

$$[\varphi, \psi] = \{v \in C^1(M) \colon \varphi \le v \le \psi \text{ on } M\}.$$

The ordinary super- and sub-solution method asserts that $(1)_\lambda$ has a solution $u_\lambda \in [\varphi, \psi]$. Further variational considerations as in [C] permits one to assume that u_λ is I_λ-minimizing in the interval $[\varphi, \psi]$, i.e.,

$$I_\lambda(u_\lambda) = \inf\{I_\lambda(v) \colon v \in [\varphi, \psi]\}. \tag{4}$$

Next, we note that there exist functions $w \in C^1(M)$ such that $I_\lambda(w) < I_\lambda(u_\lambda)$. Indeed, since $\lambda > 0$, the set $M_\varepsilon = \{x \in M \colon K_\lambda(x) > \varepsilon\}$ for small $\varepsilon > 0$ is nonempty and open. Let $f \in X$ be any function which is positive in M_ε and vanishes on $M\backslash M_\varepsilon$. Then

$$\begin{aligned} I_\lambda(tf) &= t^2 \int_M |\nabla f|^2 dv + t \int_M kf dv - \int_{M_\varepsilon} K_\lambda e^{2tf} dv - \int_{M\backslash M_\varepsilon} K_\lambda dv \\ &\le At^2 + Bt + C - \varepsilon \int_{M_\varepsilon} e^{2tf} dv \\ &\le At^2 + Bt + C - a\varepsilon e^{2ta^{-1}\int_{M_\varepsilon} f\, dv} \to -\infty, \quad \text{as } t \to \infty, \end{aligned}$$

where a is the area of M_ε. Thus, we may take $w = tf$ with t big enough. Now, if the functional satisfies the Palais-Smale condition, a result of K. C. Chang [C] asserts the existence of a mountain-pass critical point v_λ other than u_λ. The fact that I_λ does satisfy the Palais-Smale condition is proved in the next lemma. This completes the proof of (b).

Lemma 2. *Assume that the set* $M_- = \{x \in M : K_\lambda(x) < 0\}$ *is nonempty. Then the functional* I_λ *satisfies the Palais-Smale condition in the function space* $X = W^{1,2}(M)$. *That is to say, if* $\{u_k\}$ *is any sequence in* X *such that* $I_\lambda(u_k) \to c$ *for some* $c \in R$ *and* $I'_\lambda(u_k) \to 0$ *in* X^* *(the dual space of* X*), then a subsequence of* $\{u_k\}$ *converges in* X.

Proof. Let $\{u_k\}$ be the sequence in the lemma. Then we have

$$I_\lambda(u_k) = \int_M (|\nabla u_k|^2 + 2ku_k - K_\lambda e^{2u_k})\, dv \to c, \tag{5}$$

and

$$I'_\lambda(u_k)(\varphi) = \int_M (\nabla u_k \cdot \nabla\varphi + k\varphi - K_\lambda e^{2u_k}\varphi)\, dv = o(\|\varphi\|), \quad \forall \varphi \in X, \tag{6}$$

where $\|\cdot\|$ is the norm of X. Let $u_k^+ = \max\{u_k, 0\}$. We claim that $\{u_k^+\}$ is locally $W^{1,2}$-bounded in the open set M_-. More precisely, we will prove that for any domain $\Omega \subset M_-$ with $\operatorname{dist}(\Omega, \partial M_-) = d(\Omega) > 0$, we have $\|u_k^+\|_{W^{1,2}(\Omega)} \le C$, where the constant C depends only on $d(\Omega)$. To see that our claim holds it suffices to show that for any $p \in M_-$ with $\operatorname{dist}(p, \partial M_-) = d$, we have

$$\int_{B_{d/4}} (|\nabla u_k^+|^2 + (u_k^+)^2)\, dv \le C, \tag{7}$$

where B_r denote the geodesic ball centered at p with radius $r > 0$, and the constant $C > 0$ depends only on the distance d. To prove (7), let η be a smooth cut-off function supported in $B_{d/2} = B_{d/2}(p)$, such that $\eta(x) = 1$ for $x \in B_{d/4}$, $\eta(x) = 0$ for $x \in M \backslash B_{d/2}$ and $0 \le \eta \le 1$, $|\nabla\eta| \le Ad^{-1}$ on M. Substituting $\varphi = \eta^2 u_k^+$ in (6) we get

$$\int_{B_{d/2}} (\nabla u_k^+ \cdot \nabla(\eta^2 u_k^+) + k\eta^2 u_k^+ - K_\lambda e^{2u_k^+}\eta^2 u_k^+)\, dv \le C\|\eta^2 u_k^+\| \le C\|\eta u_k^+\|. \tag{8}$$

Here and in the sequel we use C to denote various constants depending only on d. Using

$$\nabla u_k^+ \cdot \nabla(\eta^2 u_k^+) = |\nabla(\eta u_k^+)|^2 + |\nabla\eta|^2 (u^+)^2,$$
$$K_\lambda \le -\varepsilon \text{ in } B_{d/2} \text{ for some } \varepsilon > 0, \text{ and } e^{2t} \ge t^3 \text{ for } t \in R,$$

we derive from (8) that

$$\int_{B_{d/2}} (|\nabla(\eta u_k^+)|^2 + \varepsilon\eta^2 (u_k^+)^4)\, dv \le -\int_{B_{d/2}} k\eta^2 u_k^+ dv + C\|\eta u_k^+\|.$$

Since $(u_k^+)^4 > (u_k^+)^2 - 1$, it is easy to see from the above inequality that

$$\varepsilon\|\eta u_k^+\|^2 \le C\|\eta u_k^+\| + C.$$

From this it follows that $\|\eta u_k^+\| \le C$, and consequently (7) holds since $\eta \equiv 1$ in $B_{d/4}$. Next, letting $\varphi \equiv 1$ in (6) we have

$$\int_M K_\lambda e^{2u_k} dv - \int_M k\, dv \to 0, \quad \text{as } k \to \infty. \tag{9}$$

Combining with (5), this gives that

$$(10) \qquad \int_M (|\nabla u|^2 + 2ku_k)\, dv = I_\lambda(u_k) - \int_M k_\lambda e^{2u_k} dv \to c - 2\pi\chi(M),$$

as $k \to \infty$.

Now we claim that $\{u_k\}$ is bounded in $L^2(M)$. If the claim is true, then (10) implies that $\{u_k\}$ is also bounded in $X = W^{1,2}(M)$. By passing to a subsequence if necessary we may assume that u_k converge weakly in X to some u_0. Then it is standard to show that u_k actually converge strongly in X using (6) and the fact that $e^{2u_k} \to e^{2u_0}$ in $L^p(M)$ for any $p \geq 1$. (Note that $\dim M = 2$.) This will finish our proof of Lemma 2.

To prove our claim we assume that on the contrary, $\|u_k\|_{L^2(M)} \to \infty$ and consider $v_k = u_k/\|u_k\|_{L^2}$, which satisfy $\|v_k\|_{L^2} = 1$ for all k. We see from (10) that

$$\int_M |\nabla v_k|^2 dv = -2 \int_M k \frac{v_k}{\|u_k\|_{L^2}} dv + o(1) \to 0.$$

It follows that v_k converges in X to some constant function $v \equiv \beta$. Since $\|v\|_{L^2} = 1$ we have $\beta \neq 0$. Note that (10) also implies that

$$\int_M k v_k dv \leq C \|u_k\|_{L^2}^{-1}.$$

Taking the limit we get that

$$\int_M \beta k dv = 2\pi\chi(M)\beta \leq 0.$$

Since β is nonzero and $\chi(M) < 0$, we must have $\beta > 0$. Now, consider $v_k^+ = u_k^+/\|u_k\|_{L^2}$. The above discussion shows that v_k^+ converge to $\beta > 0$ almost everywhere in M. However, as we have proved, u_k^+ is locally $W^{1,2}$-bounded in M_-, which implies that v_k^+ converge to 0 almost everywhere in M_-, a contradiction! This completes our proof of Lemma 2.

We now turn to

Proof of (c) *of the Theorem.* We are to prove that $(1)_{\lambda^*}$ has a solution. This will be proven by showing that certain solutions of $(1)_\lambda$ converge in X as $\lambda \to \lambda^*$.

We have seen in the proof of (b) that for $\lambda < \lambda^*$, $(1)_\lambda$ has a solution u_λ which is I_λ-minimizing in an order interval $[\varphi, \psi]$ in $C^1(M)$ (see (4)). By the maximum principle, we must have $\varphi < u_\lambda < \psi$. This implies that u_λ is a local minima for I_λ in $C^1(M)$. It follows that the second variation of I_λ at u_λ is nonnegative, i.e.,

$$(11) \qquad \int_M (|\nabla\varphi|^2 - 2K_\lambda e^{2u_\lambda}\varphi^2) dv \geq 0,$$

where $\varphi \in C^1(M)$. We also note that there is a $C > 0$ such that for $\lambda \in (0, \lambda^*)$

$$(12) \qquad u_\lambda \geq -C, \quad \text{on } M.$$

Actually, let $\varphi_c = f - c$ be the family of functions in Fact (i). Then for $c \geq$ some c_0, φ_c is a continuous family of subsolutions for $(1)_0$, hence it is also a continuous family of subsolutions for $(1)_\lambda$, where $\lambda \in (0, \lambda^*)$. We claim that

$u_\lambda \geq \varphi_{c_0}$, and consequently (12) holds. For otherwise, by varying $c \in [c_0, \infty)$, we find that for some c we have

$$u_\lambda \geq \varphi_c \text{ on } M, \text{ and } u_\lambda(x_0) = \varphi_c(x_0) \text{ for some } x_0 \in M.$$

This, by the maximum principle, can occur only if $u_\lambda \equiv \varphi_c$, which is impossible. So we see that (12) holds.

The crucial point of this proof is to show that u_λ is uniformly bounded in X as $\lambda \to \lambda^*$. If this is true, then by elliptic L^p-estimate for the solutions of $(1)_\lambda$ we see that u_λ is uniformly bounded in $W^{2,p}(M)$ for any $p > 1$. The Sobolev imbedding theorem together with Schauder estimates then imply that u_λ is uniformly $C^{2,\alpha}$-bounded. It follows that some subsequence of u_λ converges in C^2 to a solution of λ^*. This will complete our proof. We now proceed to prove the $W^{1,2}$-boundedness of u_λ. To this end we need to use the conformal invariance of equation (1). Note that u_λ being a solution of $(1)_\lambda$ is equivalent to the Gaussian curvature of $g_\lambda = e^{2u_\lambda} g$ being K_λ. If $g' = e^{2v} g$ is any metric conformal to g, then we have $g_\lambda = e^{2(u_\lambda - v)} g'$. This means that the function $w_\lambda = u_\lambda - v$ solves

$$\Delta_{g'} w - k_{g'} + K_\lambda e^{2w} = 0, \tag{13}$$

where $\Delta_{g'}$ and $k_{g'}$ are respectively the Laplacian and Gaussian curvature of g'.

Claim. The set $M^*_- = \{x \in M : K_{\lambda^*}(x) < 0\}$ is nonempty. We choose g' in (13) to be the uniqueness metric $g_0 = e^{2v_0} g$ which has constant curvature $k_0 \equiv -1$, where v_0 is the unique solution of $\Delta v - k - e^{2v} = 0$. Then $w_\lambda = u_\lambda - v_0$ is a solution of

$$\Delta_0 w + 1 + K_\lambda e^{2w} = 0. \tag{14}$$

Here and in the sequel, by the subscript $_0$ we mean that the corresponding geometric objects are for the metric g_0. Multiplying (14) by e^{-2w_λ} and integrating over M we get

$$\int_M K_\lambda dv_0 = -\int_M (2|\nabla w_\lambda|_0^2 + 1) e^{-2w_\lambda} dv_0.$$

Letting $\lambda \to \lambda^*$ we see that $\int K_{\lambda^*} \leq 0$. If the Claim is false then we must have $K_{\lambda^*} \geq 0$, and consequently $K_{\lambda^*} \equiv 0$. This contradicts our assumption that K_λ are nonconstant for all λ, showing that the Claim is true.

Now, let h be a smooth function which vanishes outside an open set D such that $\overline{D} \subset M^*_-$ and $h < 0$ in D. As in the proof of (b) of the Theorem, one may derive that u^+_λ is uniformly bounded in $W^{1,2}(D)$ for $\lambda \in (0, \lambda^*)$, and hence by a variant of the Moser-Trudinger inequality (see [C–Y2, p. 271]) we have

$$\int_D e^{2u_\lambda} \leq C. \tag{15}$$

Next, let $g_1 = e^{2v_1} g$ be the metric with Gaussian curvature h, where v_1 is the unique solution of the equation $\Delta v - k + he^{2v} = 0$. Then the function $w_\lambda = u_\lambda - v_1$ satisfies the equation

$$\Delta_1 w_\lambda - h + K_\lambda e^{2w_\lambda} = 0.$$

Since $\Delta_1 = e^{-2v_1}\Delta$, we have

$$\Delta w_\lambda - he^{2v_1} + K_\lambda e^{2(w_\lambda+v_1)} = 0. \tag{16}$$

Multiplying (16) by e^{2w_λ} and integrating over M gives

$$2\int_M |\nabla e^{w_\lambda}|^2\, dv + \int_M he^{2v_1}e^{2w_\lambda}\, dv - \int_M K_\lambda e^{2v_1}e^{4w_\lambda} dv = 0. \tag{17}$$

On the other hand, letting $\varphi = e^{w_\lambda}$ in (11) we have

$$\int_M |\nabla e^{w_\lambda}|^2\, dv - 2\int_M K_\lambda e^{2v_1}e^{4w_\lambda} \geq 0.$$

Together with (17) this gives

$$\int_M |\nabla e^{w_\lambda}|^2\, dv \leq -\frac{2}{3}\int_M he^{2(w_\lambda+v_1)}\, dv = -\frac{2}{3}\int_D he^{2u_\lambda}\, dv.$$

Thus, by (15), $|\nabla e^{w_\lambda}|$ is uniformly bounded in $L^2(M)$. We claim that $\|e^{w_\lambda}\|_{L^2(M)}$ is uniformly bounded too, consequently e^{w_λ} is uniformly bounded in X. In fact, if this is not true, we may assume that $\|e^{w_\lambda}\|_{L^2} \to \infty$ as $\lambda \to \lambda^*$. Set

$$v_\lambda = e^{w_\lambda}/\|e^{w_\lambda}\|_{L^2}.$$

Then we have

$$\|v_\lambda\|_{L^2} = 1, \quad \text{and} \quad \|\nabla v_\lambda\|_{L^2} \to 0.$$

It follows that v_λ converges in X to a constant function v with $\|v\|_{L^2} = 1$. However, (15) implies that $\|v_\lambda\|_{L^2(D)} \to 0$ as $\lambda \to \lambda^*$, and hence $v \equiv 0$ in D. But, v is constant on M, so $v \equiv 0$ on M, contradicting $\|v\|_{L^2} = 1$. This proves that e^{w_λ} and also e^{u_λ} are uniformly bounded in L^2. Actually, e^{u_λ} is uniformly L^p-bounded for any $p > 1$ since it is bounded in X.

Now we observe that since u_λ is bounded below by (12), the L^p-boundedness of e^{u_λ} implies the L^p-boundedness of u_λ. Therefore, the elliptic L^p and Schauder estimates for the solutions of $(1)_\lambda$ lead to a uniform $C^{2,\alpha}$-bound for u_λ. It follows that some subsequence of u_λ converges in C^2 to a solution of $(1)_{\lambda^*}$. This completes the proof of (c) of the Theorem.

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